Positivity of the boundedly rational agent's gain induces a potential function predicting the direction of change

Takuma Tanaka
Graduate School of Data Science, Shiga University, 1-1-1 Banba, Hikone, Shiga 522-8522, Japan
tanaka.takuma@gmail.com

Abstract

An infinite-horizon agent with a risk-neutral or risk-averse utility function trades capital and produces a consumption good. The price of non-depreciating capital, land, changes over time, and that of depreciating capital, bonds, is constant. The agent boundedly rationally chooses an action sequence that is better than the sequence maintaining a constant consumption level. It is proven that the above conditions lead to an inequality that expresses that the agent never loses from trading land. In other words, no one can outperform a boundedly rational agent by controlling the land price. This inequality is independent of the utility and production functions and the discount factor. The inequality allows us to define a thermodynamic function-like potential that predicts the direction of change of two or more interacting agents' states and suggests an analogy with phase transition and metastable states.

Keywords: Stochastic model, intertemporal choice, capital, planning, potential, metastability

JEL: D15, D25

1. Introduction

What happens when individuals act for their own interests is a key question in understanding society. The Pareto front, the Nash equilibrium, and the folk theorem are answers to this question. These concepts are useful for understanding a wide range of phenomena because they allow the prediction of what can and cannot happen, *i.e.*, the imposition of constraints on what happens. This paper aims to find a constraint for agents trading capital and producing a consumption good.

Let us assume an infinite-horizon agent whose utility is defined by a risk-neutral or risk-averse utility function depending only on the consumption in each period. In each period, the agent has a state, which includes the amounts of capital and prices. The model has both depreciating capital and non-depreciating capital, which will be referred to as bonds and land, respectively. The depreciation rate of bonds can take on any real value, so bonds can exhibit positive and negative returns. Let us assume that the land price changes over time and that the bond price does not.

The agent is assumed to know the probabilities of future events. In addition, let us assume that the agent can choose an action sequence that maintains a constant consumption level, which is referred to as a constant sequence, and that the agent boundedly rationally chooses an action sequence with greater expected utility than the constant sequence. An action sequence is a constant sequence if it does not trade land and maintains a constant amount of bonds. Note that this does not mean that the agent's state is constant for a constant sequence, because the fluctuation of the land price changes the state even if the agent does not trade land.

This paper shows that these conditions lead to an inequality that is independent of the production and utility functions, as well as the discount factor. This inequality means that the agent obtains a net gain of the consumption good by trading land. Conversely, we cannot extract profit from an agent by controlling the land price.

Unlike the Pareto front, the Nash equilibrium, and the folk theorem, which hold for multiple mutually interacting agents, the inequality holds for a single agent, as well as for multiple agents. Typically, the net gain of all agents is positive. The inequality enables us to define a quantity that characterizes the possible change of states of two or more interacting agents. This quantity is a convex function of the amounts of land and bonds and corresponds to free energy in thermodynamics. This correspondence suggests an analogy to phase transition and metastable states.

2. Model and Results

The state of the agent stochastically evolves depending on the action. In the following, the random variable representing the state is denoted by $s_\cdot$. Its index denotes the period; *i.e.*, s_t is the state at period t. Let $\underline{t}$ also denote s_t. For simplicity, this paper assumes that the states are discrete, though it is trivial to extend to continuous states. The probability of a state is denoted by $p_{\underline{t}}$. Similarly, the joint probability $p_{\underline{t},\underline{t+1}}$ and the conditional probability $p_{\underline{t+1}|\underline{t}}$ are defined. Constraints for the agent such as budget constraints are implemented by setting the probability of a state transition violating the constraints to zero.

The production of the consumption good in state $\underline{t}$ and the increase in consumption accompanying the state transition from $\underline{t}$ to $\underline{t+1}$ at period t are $f_{\underline{t}}$ and $g_{t,\underline{t},\underline{t+1}} + b_{\underline{t},\underline{t+1}}$, respectively, with $g_{t,\underline{t},\underline{t+1}}$ and $b_{\underline{t},\underline{t+1}}$ corresponding to the land and bond trades. $f_{\underline{t}}$ can be any function of capital. Because, in the following, it is assumed that the agent can choose a constant sequence, the variables $f_{\underline{t}}$ and $b_{\underline{t},\underline{t+1}}$ can denote not only the production and bond trade at period t but also those at periods $t' > t$ in a constant sequence.

As an example, if the agent trades only one type of land and the state is defined by the land price, q_t, and the amount of the land the agent owns, a_t, *i.e.*, $\underline{t} = \{a_t, q_t\}$, then we have $g_{t,\underline{t},\underline{t+1}} = q_t(a_t - a_{t+1})$. Similarly, we can define $b_{\underline{t},\underline{t+1}} = rk_t - k_{t+1}$, where k_t is the amount of bonds owned by the agent and r is the interest rate or depreciation rate of the bonds. Here, q_t and r can be any real numbers. $g_{\dots}$ is dependent on t and $b_{..}$ is independent of t because the land price, but not the bond price, changes over time. Note that the agent does not have to be a price-taker and that the prices of land and bonds do not have to exist. The following results also hold for an agent that can determine the land price and for an agent in an exchange economy without a market price.

The utility is defined by a strictly increasing concave (risk-averse) utility function $u(\cdot)$ of the single consumption good and the discount factor $0 < \beta < 1$. The consumption at period t is $c_{t,\underline{t},\underline{t+1}} = f_{\underline{t}} + g_{t,\underline{t},\underline{t+1}} + b_{\underline{t},\underline{t+1}}$. Thus, the expected utility at period t is

$$\sum_{0\leq\tau\leq\infty} \sum_{\underline{t+1},\dots,\underline{t+\tau+1}} \beta^\tau p_{\underline{t+1},\dots,\underline{t+\tau+1}|\underline{0},\cdots,\underline{t}} u(c_{t+\tau,\underline{t+\tau},\underline{t+\tau+1}}). \tag{1}$$

Because the agent can choose a constant sequence, the agent can fix the consumption as

$$\hat{c}_{t+\tau,\underline{t}} = f_{\underline{t}} + g_{t+\tau,\underline{t},\underline{t}} + b_{\underline{t},\underline{t}} \tag{2}$$

for any $\tau \geq 0$. In the following, we assume that $g_{t+\tau,\underline{t},\underline{t}} = 0$, which makes the r.h.s. of Eq. (2) constant. Let us define $\hat{u}'_{\underline{t}} = u'(\hat{c}_{t+\tau,\underline{t}})$, which is independent of τ.

To guarantee convergence, we need to assume that the limit superior and limit inferior of

$$\sum_{\underline{t+1},\dots,\underline{t+\tau+1}} p_{\underline{t+1}\cdots,\underline{t+\tau+1}|\underline{0},\cdots,\underline{t}} |x_{t+\tau,\underline{t+\tau},\underline{t+\tau+1}}| \tag{3}$$

are finite in the limit of $\tau \to \infty$, where $x_{\dots}$ is $f_{.}$, $g_{\dots}$, or $b_{..}$. Equation (3) is bounded if the state converges or there exists a steady-state distribution. In other words, growth models are excluded from our consideration. If Eq. (3) is bounded,

$$\sum_{0\leq\tau\leq\infty} \sum_{\underline{t+1},\dots,\underline{t+\tau+1}} \beta^\tau p_{\underline{t+1},\cdots,\underline{t+\tau+1}|\underline{0},\cdots,\underline{t}} x_{t+\tau,\underline{t+\tau},\underline{t+\tau+1}} \tag{4}$$

converges absolutely, and therefore the summation can be rearranged.

Because the agent chooses an action sequence that yields a greater expected utility than a constant sequence,

$$\begin{aligned} \sum_{0\leq\tau\leq\infty} \sum_{\underline{t+1},\dots,\underline{t+\tau+1}} & \beta^\tau p_{\underline{t+1},\cdots,\underline{t+\tau+1}|\underline{0},\cdots,\underline{t}} \\ & \times \{u(f_{\underline{t+\tau}} + g_{t+\tau,\underline{t+\tau},\underline{t+\tau+1}} + b_{\underline{t+\tau},\underline{t+\tau+1}}) \\ & \quad - u(f_{\underline{t}} + b_{\underline{t},\underline{t}})\} \geq 0. \end{aligned} \tag{5}$$

The concavity of the utility function gives

$$\begin{aligned}\sum_{0\le\tau\le\infty}\sum_{\underline{t+1},\ldots,\underline{t+\tau+1}}&\beta^\tau p_{\underline{t+1},\cdots,\underline{t+\tau+1}|\underline{0},\cdots,\underline{t}}\hat{u}'_{\underline{t}}\\&\times(f_{\underline{t+\tau}}+g_{t+\tau,\underline{t+\tau},\underline{t+\tau+1}}\\&\quad+b_{\underline{t+\tau},\underline{t+\tau+1}}-f_{\underline{t}}-b_{\underline{t},\underline{t}})\ge 0.\end{aligned}\tag{6}$$

Considering $\hat{u}'_{\underline{t}}>0$ and summing Eq. (6), we obtain

$$\begin{aligned}S_T=&\sum_{0\le t\le T}(1-\beta\delta_{0t})^{-1}\sum_{0\le\tau\le\infty}\sum_{\underline{0},\ldots,\underline{t+\tau+1}}\beta^\tau p_{\underline{0},\cdots,\underline{t+\tau+1}}\\&\times(f_{\underline{t+\tau}}+g_{t+\tau,\underline{t+\tau},\underline{t+\tau+1}}+b_{\underline{t+\tau},\underline{t+\tau+1}}-f_{\underline{t}}-b_{\underline{t},\underline{t}})\\=&\sum_{0\le t\le T}\sum_{\underline{t},\underline{t+1}}\frac{1}{1-\beta}p_{\underline{t},\underline{t+1}}(f_{\underline{t}}+g_{t,\underline{t},\underline{t+1}}+b_{\underline{t},\underline{t+1}}-f_{\underline{t}}-b_{\underline{t},\underline{t}})\\&+\sum_{T+1\le t\le\infty}\sum_{\underline{t},\underline{t+1}}\frac{\beta^{t-T}}{1-\beta}p_{\underline{t},\underline{t+1}}(f_{\underline{t}}+g_{t,\underline{t},\underline{t+1}}+b_{\underline{t},\underline{t+1}})\\&-\frac{\beta}{(1-\beta)^2}\sum_{\underline{0}}p_{\underline{0}}(f_{\underline{0}}+b_{\underline{0},\underline{0}})\\\ge&0.\end{aligned}\tag{7}$$

Here, $(1-\beta)^{-1}$ is decomposed into $\beta/(1-\beta)+1$, and $\sum_{0\le t\le T}\sum_{0\le\tau\le\infty}$ is replaced with $\sum_{0\le t'\le T}\sum_{0\le t\le t'}+\sum_{T+1\le t'\le\infty}\sum_{0\le t\le T}$, where $t'=t+\tau$. Defining

$$c_\infty=\liminf_{t\to\infty}\sum_{\underline{t},\underline{t+1}}p_{\underline{t},\underline{t+1}}(f_{\underline{t}}+g_{t,\underline{t},\underline{t+1}}+b_{\underline{t},\underline{t+1}}),\tag{8}$$

we find

$$\begin{aligned}\liminf_{T\to\infty}(1-\beta)S_T=&\sum_{0\le t\le\infty}\sum_{\underline{t},\underline{t+1}}p_{\underline{t},\underline{t+1}}(g_{t,\underline{t},\underline{t+1}}+b_{\underline{t},\underline{t+1}}-b_{\underline{t},\underline{t}})\\&+\frac{\beta}{1-\beta}\left(c_\infty-\sum_{\underline{0}}p_{\underline{0}}(f_{\underline{0}}+b_{\underline{0},\underline{0}})\right)\ge 0.\end{aligned}\tag{9}$$

Let us now examine a few special cases. Assume that there are one kind of land and one kind of bonds. Then, we can define

$$g_{t,i,i'}=q_{t,i}(a_i-a_{i'})+\gamma^g_{ti}(a_i-a_{i'}),\tag{10}$$
$$b_{i,i'}=rk_i-k_{i'}+\gamma^b_i(k_i-k_{i'}),\tag{11}$$

where $q_{t,i}$ is the land price at period t and r is the interest rate of the bonds. $\gamma^g_{tij}(\cdot)$ and $\gamma^b_{ij}(\cdot)$ are the costs of adjustment for land and bonds, respectively. Under the assumption that the state distribution converges to the $t=0$ distribution, the second term of Eq. (9) vanishes if the cost of adjustment vanishes. For example, $\gamma^g_{tij}(x) = \gamma^b_{ij}(x) = x^2$ and the state changes infinitesimally slowly. If the agent returns to the same steady state as the initial condition, rearranging the first term by using

$$\sum_{0\le t\le T}\sum_{\underline{t},\underline{t+1}} p_{\underline{t},\underline{t+1}}(b_{\underline{t},\underline{t+1}} - b_{\underline{t},\underline{t}}) = \sum_{\underline{0}} p_{\underline{0}}k_{\underline{0}} - \sum_{\underline{T+1}} p_{\underline{T+1}}k_{\underline{T+1}}, \tag{12}$$

yields

$$\sum_{0\le t\le\infty}\sum_{\underline{t},\underline{t+1}} p_{\underline{t},\underline{t+1}}g_{t,\underline{t},\underline{t+1}} \geq 0; \tag{13}$$

i.e., the agent will not have negative profit. This inequality also holds if the agent can choose between fixing the production function and drawing it from a distribution, which can be regarded as a model of job search. Note that Eq. (9) holds even if the land price keeps changing.

Another special case is a risk-neutral agent with identically and independently distributed production functions. We decompose the state into two components, $s_. = (\kappa_., \lambda_.)$, where $\lambda_.$ denotes the production function and $\kappa_.$ represents the other elements of the state, such as the amounts of capital owned by the agent and their prices. By stopping all trading of capital, the agent can maintain $\kappa_.$ at a constant value. If the agent can choose an action sequence with a greater expected utility than a constant sequence, then we have

$$\begin{aligned}\sum_{0\le\tau\le\infty}\sum_{\underline{t+1},\ldots,\underline{t+\tau+1}} &\beta^\tau p_{\kappa_{t+1},\lambda_{t+1},\cdots,\kappa_{t+\tau+1},\lambda_{t+\tau+1}|\underline{t}} \\ &\times (f_{\kappa_{t+\tau},\lambda_{t+\tau}} + g_{t+\tau,\kappa_{t+\tau},\kappa_{t+\tau+1}} + b_{\kappa_{t+\tau},\kappa_{t+\tau+1}} \\ &\quad - f_{\kappa_t,\lambda_{t+\tau}} - b_{\kappa_t,\kappa_t}) \geq 0.\end{aligned} \tag{14}$$

If the production function at each period is drawn from an identical and independent distribution,

$$\begin{aligned}&\sum_{0\le t\le\infty}\sum_{\kappa_t,\lambda_t,\kappa_{t+1},\lambda_{t+1}} p_{\kappa_t,\lambda_t,\kappa_{t+1}\lambda_{t+1}}(g_{t,\kappa_t,\kappa_{t+1}} + b_{\kappa_t,\kappa_{t+1}} - b_{\kappa_t,\kappa_t}) \\ &+ \frac{\beta}{1-\beta}\left(c_\infty - \sum_{\kappa_0,\lambda_0} p_{\kappa_0,\lambda_0}(f_{\kappa_0,\lambda_0} + b_{\kappa_0,\kappa_0})\right) \geq 0\end{aligned} \tag{15}$$

holds, which also means that the agent will not have negative profit. Specifically, if the distribution of states as $t \to \infty$ converges to the initial distribution of states, the land price is not stochastic, and there is no cost of adjustment, then we have

$$\sum_{0\le t\le\infty}\sum_{\kappa_t,\kappa_{t+1}} p_{\kappa_t,\kappa_{t+1}}g_{t,\kappa_t,\kappa_{t+1}} \geq 0. \tag{16}$$

3. Discussion

This paper has shown that if the expected value of the consumption converges to the initial value and the cost of adjustment vanishes, then a risk-neutral agent with uncertainty in production and a risk-neutral and risk-averse agent without uncertainty in production both make profits from trading land. Specifically, Eq. (9) holds for cases of (a) the land price changing stochastically, (b) a cost of adjustment existing, (c) the agent being able to choose between keeping or changing its production function, and (d) the agent being risk neutral and the production function and land price changing independently. The amounts of capital the agent is allowed to own can be any subset of $\mathbb{R}^C$, where C is the number of kinds of capital. The present results are extensions of previous results (Tanaka, 2020, 2022).

Although the present study has assumed that utility depends only on the consumption at each period, the results are applicable to some models including labor in the utility function. For example, the utility function of the model (Bond and Park, 2002) in section 26.3.1 of Ljungqvist and Sargent (2018) is $u(c, l) = c + l - 0.5l^2$, where c is consumption and l is leisure. In this model, l satisfies $l + n_1 + n_2 = 1$, where n_1 and n_2 are the times for the production of goods 1 and 2, respectively. Consumption is given by $c = 2\min\{n_1, \gamma n_2\} + \bar{y}$, where $\bar{y}$ is the endowment. This model can be regarded as a risk-neutral agent having the production function

$$\max_{l,n_1,n_2} \ 2\min\{n_1, \gamma n_2\} + l - 0.5l^2 + \bar{y} \quad \text{s.t. } l + n_1 + n_2 = 1, \tag{17}$$

if the trade of land and bonds is included in $\bar{y}$.

Although this paper has assumed that the agent knows the probabilities of future events, this assumption can be relaxed. If the agent can choose the pair of an action at $t = \tau$ and a constant sequence for $t \geq \tau + 1$ with a greater utility than the constant sequence for $t \geq \tau$ (Tanaka, 2020), then the agent does not need to know the probabilities of future events for Eq. (13) to hold.

Determining whether the present results hold for a risk-averse agent with a stochastic production function remains as future work. Such a case will require additional assumptions. For the linear-quadratic approximation such as used in dynamic stochastic general equilibrium (DSGE) models, the present results hold under the assumption that the stochastic component is sufficiently small, *i.e.*, under linear approximation with $u'(c_{\underline{t}}) > 0$.

We should note that the nonnegativity of the agent's profit leads to an analogy to thermodynamics. For simplicity, let us make the following assumptions: there is only one kind of capital (land); its amount can take on any real value; the land price is fixed at each period; there is no cost of adjustment; the production function does not change; and the agent is risk neutral or risk averse and chooses the optimal action. If we infinitesimally slowly change the land price and restore its initial value, the agent's gain is zero because Eq. (13) holds for any trajectory and its reverse. Similarly to the Helmholtz free energy derived from Kelvin's principle (Tanaka, 2022; Tasaki and Paquette, unpublished), a potential at state x can be defined by the agent's minimal gain obtained in a trajectory starting from an arbitrary reference state and ending in state x. The sum of the potentials of agents decreases when they trade land and converge to a steady state. This is because if the sum increased, one could gain from these agents by restoring their initial states in trading with them, which contradicts Eq. (13). Thus, the potential predicts the possible direction of change. The convexity of the potential can be proven by considering the process in which $n^{(1)}$ agents with state $s^{(1)}$ and $n^{(2)}$ agents with state $s^{(2)}$ become $n^{(1)} + n^{(2)}$ agents with the identical state $(n^{(1)}s^{(1)} + n^{(2)}s^{(2)})/(n^{(1)} + n^{(2)})$ in trading capital (Tanaka, 2022; Tasaki and Paquette, unpublished).

This analogy leads us to metastability and first-order phase transition. If the land price q is constant, differentiating Eq. (1) with respect to the amount of land a yields

$$-qu'(c) + \beta \left(\frac{\partial f}{\partial a} + q \right) u'(c) = 0, \tag{18}$$

which holds at the equilibrium. The equilibrium is stable if $-q + \beta(f' + q)$ is decreasing with respect to a. If the production function is convex at the point, it is unstable and never realized. Assume that Eq. (18) has exactly two distinct stable solutions, a' and a''. If $a' < a''$, then an agent with land a' cannot necessarily change the amount of land from a'' if q decreases only a little. The amount of land suddenly exceeds a'' if q becomes small enough. This can be regarded as a metastable state in a first-order phase transition. If there are an infinite number of agents, then the agents with a' and a'' can be mixed in any fraction. This corresponds to a macroscopic state in which the amount of land lies between a' and a''.

If the production function smoothly changes with a parameter, then phase transition and critical phenomena can appear. If the production function is continuously differentiable, then the Landau theory holds, and the critical exponents coincide with the classical ones (Tasaki and Paquette, unpublished). However, the critical exponents might take on a different value if an infinite number of agents interact with each other. Moreover, the model in which the second term of Eq. (9) does not vanish may correspond to non-equilibrium thermodynamics with a constant dissipation.

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