Predicting agents' direction of change in a non-stochastic economy using a potential function

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Abstract

This study examines an optimal agent producing a consumption good and depreciating capital and trading capital without depreciation. Assuming that the prices of the depreciating capital are fixed and that the future prices of undepreciated capital are announced, this study demonstrates that the agent never loses profit on trading undepreciated capital if the agent's state converges to the initial state. Corollary to this result, we found a scalar potential that predicts the change direction of agents trading the undepreciated capital exclusively among them. The similarity between the scalar potential and the Helmholtz free energy suggests that stochastic economic models could be characterized by a framework similar to information thermodynamics.

Keywords: Non-stochastic model, intertemporal choice, capital, planning

JEL: D15, D25

1. Introduction

The characterization of an agent's optimal decision-making (Bertsekas, 2005) without using the agent's detailed information is of great value. This type of characterization would allow for a prediction applicable to any agent. The amount of goods traded by the agent is a characteristic that can be measured without the detailed information. Thus, in this note, we focus on the total quantity traded by an agent, as a step toward the characterization of optimal decision-making. We investigate a model in which an imaginary government can control the price and supply of land, water rights, and fishery rights, that is, the capital without depreciation, of the region where the agent is living. This note demonstrates whether the government can gain profit from trading the undepreciated capital at previously announced prices.

This note is organized as follows. Section 2 describes the model and states the main result; that is, the government never gains profit. Section 3 sketches the implication of the result. We obtain a function that predicts the direction of change of agents trading the undepreciated capital exclusively among them. Section 4 summarizes the results and suggests future directions. Finally, Appendix A presents the proof of the main result.

2. Model and the main result

In this model, an agent produces a consumption good and depreciating capital, and the agent trades capital without depreciation. For brevity, we refer to depreciating capital simply as "capital" and to capital without depreciation as "land." We assume that a unit of consumption good can be exchanged with fixed amounts of capital. In other words, the agent may trade the consumption good and capital in a large market, in which their prices are constant. In contrast, the prices of land are not fixed. An entity referred to as government announces the time series of the land prices and trades the land with the agent. The government prohibits the agent from selling land to the large market. This model does not consider stochasticity.

The capital and land owned by the agent in period t are denoted by $\boldsymbol{k}_t \in \mathbb{R}^{N_{\text{capital}}}$ and $\boldsymbol{a}_t \in \mathbb{R}^{N_{\text{land}}}$, respectively. For the land price vector, the government announces its sequence in period $t = 0$, measured in the unit of the consumption good in period t, $\boldsymbol{q}_t$, and it is known to the agent. Meanwhile, the price vector of the capital is fixed to $\boldsymbol{r}$. The agent maximizes

$$\sum_{0 \le t \le \infty} \beta^t u[f(\boldsymbol{a}_t, \boldsymbol{k}_t) - \boldsymbol{q}_t \cdot (\boldsymbol{a}_{t+1} - \boldsymbol{a}_t) - \boldsymbol{r} \cdot (\boldsymbol{k}_{t+1} - \boldsymbol{\rho} \circ \boldsymbol{k}_t)] \tag{1}$$

with $\boldsymbol{a}_0$ and $\boldsymbol{k}_0$ given at $t = 0$, where $\circ$ is the element-wise product. The utility function $u(\cdot)$ and the production function $f(\cdot)$ are concave. The discount factor β satisfies $0 < \beta < 1$. ρ_i is defined by $1 - \delta_i$, where $0 < \delta_i < 1$ is the depreciation rate of capital i.

Appendix A demonstrates that

$$\sum_{0 \le t \le \infty} \boldsymbol{q}_t \cdot (\boldsymbol{a}_{t+1} - \boldsymbol{a}_t) \le 0 \tag{2}$$

if the state of the agent converges to the initial state in the limit of $t \to \infty$ (*i.e.*, $\lim_{t\to\infty} \boldsymbol{a}_t = \boldsymbol{a}_0$ and $\lim_{t\to\infty} \boldsymbol{k}_t = \boldsymbol{k}_0$). This means that the government cannot gain profit from trading land at the previously announced prices, making the agent's final state identical to its initial state. Here we refer to $\boldsymbol{a}$ as the agent's state because $\boldsymbol{k}$ is determined by $\boldsymbol{a}$ at the steady state if $f(\cdot)$ is a strictly concave function.

3. Implication of the main result

At the steady state, the price vector of the land is

$$\boldsymbol{q} = \frac{1}{\beta^{-1} - 1} \nabla_{\boldsymbol{a}} f. \tag{3}$$

Therefore, the net gain of the government in an infinitely slow cyclic change C of the land prices is $\oint_C \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a}$. If $\oint_C \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a} > 0$, it contradicts Eq. (2). If $\oint_C \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a} < 0$, reversing the price sequence yields $\oint_{-C} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a} > 0$, which contradicts Eq. (2). Hence,

$$\oint_C \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a} = 0 \tag{4}$$

holds for any cyclic integral. This means that

$$F(\boldsymbol{a}) = \int_{\boldsymbol{a}}^{\boldsymbol{a}_{\mathrm{ref}}} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a} = \frac{1}{\beta^{-1} - 1} \int_{\boldsymbol{a}}^{\boldsymbol{a}_{\mathrm{ref}}} \nabla_{\boldsymbol{a}} f \cdot \mathrm{d}\boldsymbol{a}, \tag{5}$$

where $\boldsymbol{a}_{\mathrm{ref}}$ is an arbitrary reference point, does not depend on the path of integration connecting $\boldsymbol{a}$ and $\boldsymbol{a}_{\mathrm{ref}}$. That is, $F(\cdot)$ is a scalar potential. $F(\boldsymbol{a})$ can be viewed as the government's profit gained by slowly changing the agent's state from $\boldsymbol{a}$ to $\boldsymbol{a}_{\mathrm{ref}}$. Moreover, the government's profit gained by slowly changing the agent's state from $\boldsymbol{a}_{\mathrm{ref}}$ to $\boldsymbol{a}$ is $-F(\boldsymbol{a})$. Meanwhile, $F(\boldsymbol{a}) - F(\boldsymbol{a}')$ is the maximal profit that the government can gain by changing the agent's state from $\boldsymbol{a}$ to $\boldsymbol{a}'$ because if the government can gain $\tilde{F} > F(\boldsymbol{a}) - F(\boldsymbol{a}')$ by changing the agent's state from $\boldsymbol{a}$ to $\boldsymbol{a}'$, it can gain $\tilde{F} + F(\boldsymbol{a}') - F(\boldsymbol{a}) > 0$ in a process in which the agent's state visits $\boldsymbol{a}$, $\boldsymbol{a}'$, $\boldsymbol{a}_{\mathrm{ref}}$, and $\boldsymbol{a}$ in sequence. This contradicts Eq. (2).

Next, we examine the convexity of $F(\boldsymbol{a})$. Let us denote the vector whose i-th element is $\frac{\partial f(\boldsymbol{a},\boldsymbol{k})}{\partial x_i}$ by $f_{\boldsymbol{x}}(\boldsymbol{a}, \boldsymbol{k})$ and the matrix whose $(i,\ j)$ element is $\frac{\partial^2 f(\boldsymbol{a},\boldsymbol{k})}{\partial x_i \partial y_i}$ by $f_{\boldsymbol{xy}}(\boldsymbol{a}, \boldsymbol{k})$. Defining $\boldsymbol{k}(\boldsymbol{a})$ by the solution of $f_{\boldsymbol{k}}(\boldsymbol{a}, \boldsymbol{k}) = \boldsymbol{b} \circ \boldsymbol{r}$, where $b_i = 1/(\beta^{-1} - \rho_i)$, yields

$$F(\boldsymbol{a}) = \frac{1}{\beta^{-1} - 1} \int_{\boldsymbol{a}}^{\boldsymbol{a}_{\mathrm{ref}}} f_{\boldsymbol{a}}[\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{a})] \cdot \mathrm{d}\boldsymbol{a}. \tag{6}$$

Its Hessian with respect to $\boldsymbol{a}$ is given by

$$\nabla_{\boldsymbol{a}} \otimes \nabla_{\boldsymbol{a}} F(\boldsymbol{a}) = -\frac{1}{\beta^{-1} - 1} \{ f_{\boldsymbol{aa}}[\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{a})] + f_{\boldsymbol{ak}}[\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{a})] \nabla_{\boldsymbol{a}} \boldsymbol{k}(\boldsymbol{a}) \}. \tag{7}$$

Because

$$\nabla_{\boldsymbol{a}} \boldsymbol{k} = -f_{\boldsymbol{kk}}(\boldsymbol{a}, \boldsymbol{k})^{-1} f_{\boldsymbol{ka}}(\boldsymbol{a}, \boldsymbol{k}), \tag{8}$$

we obtain

$$\nabla_{\boldsymbol{a}} \otimes \nabla_{\boldsymbol{a}} F(\boldsymbol{a}) = -\frac{1}{\beta^{-1} - 1} [f_{\boldsymbol{aa}}(\boldsymbol{a}, \boldsymbol{k}) - f_{\boldsymbol{ak}}(\boldsymbol{a}, \boldsymbol{k}) f_{\boldsymbol{kk}}(\boldsymbol{a}, \boldsymbol{k})^{-1} f_{\boldsymbol{ka}}(\boldsymbol{a}, \boldsymbol{k})]. \tag{9}$$

The concavity of $f(\boldsymbol{a},\boldsymbol{k})$ implies the positive definiteness of

$$\begin{pmatrix} -f_{\boldsymbol{aa}}(\boldsymbol{a},\boldsymbol{k}) & -f_{\boldsymbol{ak}}(\boldsymbol{a},\boldsymbol{k}) \\ -f_{\boldsymbol{ka}}(\boldsymbol{a},\boldsymbol{k}) & -f_{\boldsymbol{kk}}(\boldsymbol{a},\boldsymbol{k}) \end{pmatrix} \tag{10}$$

and, consequently, the positive definiteness of $-f_{\boldsymbol{aa}}(\boldsymbol{a},\boldsymbol{k}) + f_{\boldsymbol{ak}}(\boldsymbol{a},\boldsymbol{k}) f_{\boldsymbol{kk}}(\boldsymbol{a},\boldsymbol{k})^{-1} f_{\boldsymbol{ka}}(\boldsymbol{a},\boldsymbol{k})$. Combining it with Eq. (9) demonstrates the convexity of $F(\boldsymbol{a})$.

Let us investigate the interaction among more than one agent by using $F(\boldsymbol{a})$. We assume that N_{agent} agents have heterogeneous production functions, utility functions, and discount factors. For each agent i, $F^{(i)}(\boldsymbol{a}^{(i)})$ can be defined. The government can let the agents trade the land among themselves without intervention. This situation is achieved by designing the sequence of $\boldsymbol{q}_t$, so that $\sum_{1\le i\le N_{\text{agent}}}(\boldsymbol{a}_{t+1}^{(i)} - \boldsymbol{a}_t^{(i)}) = \boldsymbol{0}$ for $t \ge 0$. Let us examine what happens if the agents' states start from $\boldsymbol{a}_0^{(i)}$ and converge to $\boldsymbol{a}_\infty^{(i)}$ without government intervention and then return from $\boldsymbol{a}_\infty^{(i)}$ to $\boldsymbol{a}_0^{(i)}$ in trading the land with government intervention. If $\sum_{1\le i\le N_{\text{agent}}} F^{(i)}(\boldsymbol{a}_0^{(i)}) < \sum_{1\le i\le N_{\text{agent}}} F^{(i)}(\boldsymbol{a}_\infty^{(i)})$, the government can gain $\sum_{1\le i\le N_{\text{agent}}}[F^{(i)}(\boldsymbol{a}_\infty^{(i)}) - F^{(i)}(\boldsymbol{a}_0^{(i)})] > 0$ by its intervention to the agents. This contradicts Eq. (2). Hence,

$$\sum_{1\le i\le N_{\text{agent}}} F^{(i)}(\boldsymbol{a}_0^{(i)}) \ge \sum_{1\le i\le N_{\text{agent}}} F^{(i)}(\boldsymbol{a}_\infty^{(i)}). \tag{11}$$

This means that trading the land exclusively among agents never increases the summation of $F^{(i)}(\boldsymbol{a}^{(i)})$. This argument holds even if the land is traded by only a fraction of agents for only a few periods.

4. Conclusion

This note has demonstrated that the government, which is capable of controlling the land prices as previously announced, cannot profit if the capital price is fixed and the agent's state converges to the initial state. This result holds for an agent with any utility function, production function, and discount factor. A scalar potential that predicts the direction of change in agents' states, $F(\cdot)$, is derived from the result.

This study's result is an infinite-period extension of a previous result on an exchange economy with two periods (Tanaka, 2020). The result can be extended to an agent that trades the consumption goods and capital with fixed prices and can produce them with one unit of labor endowment.

Is the present result applicable to an economy with consumption goods and capital with changing prices? Numerically, counterexamples to Eq. (2) are found for this type of economy. In other words, this type of economy allows for an agent profiting from the cyclic trade of the land. Thus, in the real economy, one possibility is that some agents gain profit by cyclic trade. Another possibility is that the real economy oscillates to prohibit agents from reaching steady states where agents can gain profit. Further work must determine which of these two possibilities the case is. Moreover, extension to stochastic models would be of interest because this extension enables us to apply the result to a broader range of settings.

This approach is different from the previous approaches to economic dynamics (Stokey *et al.*, 1989; Ljungqvist and Sargent, 2012). Moreover, this note does not give the solution to the optimization problem and the stability of the dynamics, but the condition that the solutions must satisfy. Integrating both approaches will be useful to investigate the behavior of an agent if this approach could be extended to stochastic dynamics.

The similarity of the mathematical frameworks of this model and thermodynamics is also of interest. Considering the analogy between the negativity of the government's gain and the negativity of the work done by a system in an isothermal environment, $F(\cdot)$ is the economic counterpart of the Helmholtz free energy. Helmholtz free energy is also a convex function that determines the direction of change in physical systems (Callen, 1985). Recently, the behavior of stochastic systems has been described by information thermodynamics (Ito and Sagawa, 2013). Thus, stochastic economic models could be characterized by a framework similar to information thermodynamics.

A. Proof

We demonstrate the theorem for the case of $N_{\mathrm{land}} = N_{\mathrm{capital}} = 1$ to simplify the proof, although extending the derivation to the case of $N_{\mathrm{land}} > 1$ and $N_{\mathrm{capital}} > 1$ is trivial. Accordingly, we replace $\boldsymbol{a}_t$, $\boldsymbol{k}_t$, $\boldsymbol{q}_t$, $\boldsymbol{r}$, $\boldsymbol{\rho}$ with a_t, k_t, q_t, r, and ρ, respectively, in the following. The necessary conditions for the maximization of Eq. (1) are

$$u'_t \frac{\partial}{\partial a_t} f(a_t, k_t) + u'_t q_t - \beta^{-1} u'_{t-1} q_{t-1} = 0, \tag{12}$$

$$u'_t \frac{\partial}{\partial k_t} f(a_t, k_t) + \rho u'_t r - \beta^{-1} u'_{t-1} r = 0 \tag{13}$$

for $t \geq 1$, where we defined

$$u'_t = \left. \frac{\partial u(c)}{\partial c} \right|_{c = f(a_t, k_t) - q_t(a_{t+1} - a_t) - r(k_{t+1} - \rho k_t)}. \tag{14}$$

We do not assume the nonnegativity of the amount of capital, land, and consumption. However, Inada conditions can guarantee the nonnegativity.

Using Eqs. (12) and (13) leads the concavity of $u(\cdot)$ and $f(\cdot)$ to

$$\begin{aligned} x_{t,s} =& u'_t f_t - u'_t f_s + (u'_t q_t - \beta^{-1} u'_{t-1} q_{t-1})(a_t - a_s) \\ &+ (\rho u'_t r - \beta^{-1} u'_{t-1} r)(k_t - k_s) \geq 0, \end{aligned} \tag{15}$$

$$\begin{aligned} y_{t,s} =& u_t - u_s - u'_t [f_t - q_t(a_{t+1} - a_t) - r(k_{t+1} - \rho k_t) \\ &- f_s + q_s(a_{s+1} - a_s) + r(k_{s+1} - \rho k_s)] \geq 0, \end{aligned} \tag{16}$$

where

$$f_t = f(a_t, k_t), \tag{17}$$

$$u_t = u[f(a_t, k_t) - q_t(a_{t+1} - a_t) - r(k_{t+1} - \rho k_t)]. \tag{18}$$

In the following, we show

$$\lim_{T \to \infty} \sum_{0 \leq i \leq T} z_i = - \sum_{0 \leq t \leq \infty} q_t(a_{t+1} - a_t), , \tag{19}$$

where

$$w_i = \begin{cases} \left((1-\beta)\sum_{t=0}^{\infty}\beta^t u'_{t+i}\right)^{-1} & i=0 \\ \left(\sum_{t=0}^{\infty}\beta^t u'_{t+i}\right)^{-1} & i\geq 1 \end{cases}, \tag{20}$$

$$z_i = w_i \sum_{i+1\leq t\leq\infty}\beta^{t-i}x_{t,i} + w_i(1-\beta)\sum_{\substack{i\leq s\leq\infty\\ i\leq t\leq\infty}}\beta^{s+t-2i}y_{t,s}. \tag{21}$$

Equation (19) proves Eq. (2) because the left-hand-side of Eq. (19) is the summation of the nonnegative values $x_{t,s}$ and $y_{t,s}$ with nonnegative weights.

Combining

$$\begin{aligned} &\sum_{i+1\leq t\leq\infty}\beta^{t-i}x_{t,i} \\ =&\sum_{i+1\leq t\leq\infty}\beta^{t-i}\{u'_t f_t + (u'_t q_t - \beta^{-1}u'_{t-1}q_{t-1})a_t + (\rho u'_t r - \beta^{-1}u'_{t-1}r)k_t\} \\ &-\sum_{i+1\leq t\leq\infty}\beta^{t-i}u'_t f_i + u'_i q_i a_i + u'_i r k_i + \sum_{i+1\leq t\infty}\beta^{t-i}u'_t(1-\rho)rk_i \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\sum_{\substack{i\leq s\leq\infty\\ i\leq t\leq\infty}}\beta^{s+t-2i}y_{t,s} \\ =&-\frac{1}{1-\beta}\sum_{i\leq t\leq\infty}\beta^{t-i}u'_t[f_t - q_t(a_{t+1}-a_t) - r(k_{t+1}-\rho k_t)] \\ &-\sum_{i\leq t\leq\infty}\beta^{t-i}u'_t\sum_{i\leq s\leq\infty}\beta^{s-i}[-f_s + q_s(a_{s+1}-a_s) + r(k_{s+1}-\rho k_s)], \end{aligned} \tag{23}$$

yields

$$\begin{aligned} w_i^{-1}z_i =& -\sum_{i\leq t\leq\infty}\beta^{t-i}u'_t f_i + \sum_{i\leq t\leq\infty}\beta^{t-i}u'_t(1-\rho)rk_i \\ &-(1-\beta)\sum_{i\leq t\leq\infty}\beta^{t-i}u'_t\sum_{i\leq s\leq\infty}\beta^{s-i}[-f_s + q_s(a_{s+1}-a_s) + r(k_{s+1}-\rho k_s)]. \end{aligned} \tag{24}$$

Thus,

$$z_0 = -\frac{1}{1-\beta}f_0 + \frac{1}{1-\beta}(1-\rho)rk_0 - \sum_{0\leq s\leq\infty}\beta^s[-f_s + q_s(a_{s+1}-a_s) + r(k_{s+1}-\rho k_s)] \tag{25}$$

and

$$z_i = -f_i + (1-\rho)rk_i - (1-\beta)\sum_{i\leq s\leq\infty}\beta^{s-i}[-f_s + q_s(a_{s+1}-a_s) + r(k_{s+1}-\rho k_s)] \tag{26}$$

for $i \geq 1$. Moreover,

$$\begin{aligned}\sum_{1\leq i\leq T} z_i = &- \sum_{1\leq s\leq T} q_s(a_{s+1}-a_s) + rk_1 - rk_{T+1} \\ &+ \sum_{1\leq s\leq\infty} \beta^s[-f_s + q_s(a_{s+1}-a_s) + r(k_{s+1}-\rho k_s)] \\ &- \sum_{T+1\leq s\leq\infty} \beta^{s-T}[-f_s + q_s(a_{s+1}-a_s) + r(k_{s+1}-\rho k_s)]; \end{aligned} \tag{27}$$

thus, the summation is given by

$$\begin{aligned}\sum_{0\leq i\leq T} z_i = &- \frac{\beta}{1-\beta} f_0 + \frac{1}{1-\beta}(1-\rho)rk_0 + r\rho k_0 - rk_{T+1} - \sum_{0\leq s\leq T} q_s(a_{s+1}-a_s) \\ &- \sum_{T+1\leq s\leq\infty} \beta^{s-T}[-f_s + q_s(a_{s+1}-a_s) + r(k_{s+1}-\rho k_s)]. \end{aligned} \tag{28}$$

Taking the limit of $T \to \infty$, we obtain

$$\begin{aligned}\lim_{T\to\infty} \sum_{0\leq i\leq T} z_i = &- \sum_{0\leq s\leq\infty} q_s(a_{s+1}-a_s) - \frac{\beta}{1-\beta} f_0 + \frac{1}{1-\beta}(1-\rho)rk_0 \\ &+ r\rho k_0 - rk_0 - \frac{\beta}{1-\beta}[-f_0 + r(1-\rho)k_0] \\ = &- \sum_{0\leq s\leq\infty} q_s(a_{s+1}-a_s), \end{aligned} \tag{29}$$

where we used $\lim_{t\to\infty} a_t = a_0$, $\lim_{t\to\infty} k_t = k_0$, and $\lim_{t\to\infty} f_t = f_0$. This completes the proof.

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